November 7, 2008

LBL-40890
UCB-PTH-97/50

Path Integral Quantization of the Symplectic Leaves of the $SU(2)^*$ Poisson-Lie Group[†]

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Abstract

The Feynman path integral is used to quantize the symplectic leaves of the Poisson-Lie group $SU(2)^*$. In this way we obtain the unitary representations of $\mathcal{U}_q(su(2))$. This is achieved by finding explicit Darboux coordinates and then using a phase space path integral. I discuss the $*$-structure of $SU(2)^*$ and give a detailed description of its leaves using various parametrizations. I also compare the results with the path integral quantization of spin.

[†]This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-95-14797

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Disclaimer

1 Introduction

The Feynman path integral reveals in a geometric intuitive way the relation between classical and quantum dynamics. However there are few examples of path integral quantizations on compact phase spaces. These are interesting because they have finite dimensional Hilbert spaces. The simplest example is a phase space with the topology of a torus. A more interesting case is obtained by considering a phase space with the topology of the sphere S_2. Quantization of this gives the spin. A path integral quantization is described in [1, 2]. Here I will present a generalization of this result, the case of the deformed spin.

Let G be a Lie group. On the vector space g^* dual to the Lie algebra g of G there is a natural Poisson structure. In terms of linear coordinates e_i and f_{ij}^k the structure constants of the group it has the form

$$\{e_i, e_j\} = f_{ij}^k\, e_k$$

and it is known as the Lie-Kirillov-Kostant Poisson bracket. Its symplectic leaves are the orbits of the coadjoint action [13]. The quantization of this bracket is the universal enveloping algebra $\mathcal{U}(g)$ which is the associative algebra with generators e_i and relations

$$[e_i, e_j] = i\,\hbar\, f_{ij}^k\, e_k.$$

Quantization of the coadjoint orbits of a Lie group G gives its unitary representations [13]. Various methods were used to quantize these symplectic leaves including geometric quantization and the Feynman path integral [1, 2]. Note that the vector space g^* can be thought of as an abelian group. The above picture can be generalized to include Poisson brackets on non-abelian groups G^* usually called the dual Poisson-Lie groups. This will be extensively discussed in Section 2. Quantization of their symplectic leaves gives the unitary representations of the quantum group $\mathcal{U}_q(g)$. This can be summarized

in the picture below.

$$\begin{array}{ccc} Fun(G^*) & \rightarrow & Fun_q(G^*) \cong \mathcal{U}_q(g) \\ \uparrow & & \uparrow \\ Fun(g^*) & \rightarrow & \mathcal{U}(g) \end{array}$$

The quantization axis is horizontal, with classical Poisson-Lie groups on the left and their quantizations on the right. The vertical axis corresponds to deformation of the abelian case to the non-abelian case. Note that the abelian case can be obtained from the non-abelian case by looking at an infinitesimal neighborhood of the unit of the group, and rescaling coordinates appropriately. Throughout this paper I will refer to the lower part of the picture already discussed in [1, 2] as the trivial case[§], and to the upper part as the Poisson case.

I will use the Feynman path integral to quantize the symplectic leaves of $SU(2)^*$. In doing this I will follow closely the method used in [1]. In fact, a strong parallel exists both at the classical and the quantum levels. Classically, the leaves coincide in the trivial and Poisson cases once expressed in terms of Darboux coordinates. Consequently, at the quantum level we have the same Hilbert space and the two quantum algebras are isomorphic. The path integral has the same form in the trivial and Poisson cases, but one has to insert different functions to obtain $su(2)$ or $\mathcal{U}_q(su(2))$ generators.

In Section 2, I review some general Poisson-Lie theory mainly to fix the notation and to list some results used later in the paper. The results in this section are given using complex coordinates. In Section 3, I describe the reality structures of $SU(2)$, its dual and its double. I also give a detailed description of the symplectic leaves of $SU(2)^*$.

In Section 4, I describe Darboux coordinates, formulate the path integral and find the radius quantization condition using a quantization condition similar to [1]. I also define the Hilbert space and obtain the matrix elements

[§]The Poisson bracket on G discussed in Section 2 is trivial in this case

of diagonal operators. In Section 5, I study general matrix elements and show that they are representations of the quantum group algebra. In the last section I draw some conclusions and suggest how this work might be generalized. Finally, the appendix reviews the isomorphism of $Fun_q(SU(2)^*)$ and $\mathcal{U}_q(su(2))$ and the derivation of the Poisson bracket on $SU(2)^*$ from $Fun_q(SU(2)^*)$.

2 Dual Pairs of Poisson-Lie Groups

A *Poisson-Lie Group* (PLG) is a pair $(G, \{,\})$ where G is a Lie group and $\{,\}$ is a Poisson bracket on G which is compatible with the group operations of multiplication and inversion [8]. The compatibility determines the Poisson structure at an arbitrary point from its values in the vicinity of the group unit. A PLG can be equivalently described as a *Poisson Hopf algebra* $Fun(G)$ which is a commutative Hopf algebra with a compatible Poisson algebra. In what follows I will freely exchange these two dual descriptions.

The Poisson bracket on the group determines a Lie algebra structure on the cotangent space g^* of the Lie group. Let h_1 and h_2 be two functions on the group G. Then:

$$[dh_1, dh_2]^* \equiv d\{h_1, h_2\}$$

defines a Lie algebra $(g^*, [,]^*)$. One can check that this definition is independent of the choice of functions used to represent cotangent vectors. Let $\{e_i\}$ be a basis of g, $\{e^i\}$ its dual basis in g^*, and f_{ij}^k and $\tilde{f}_c^{ab}$ the corresponding structure constants. The compatibility of the Poisson and group structures imposes restrictions on the two Lie algebras. In terms of the structure constants, they read

$$f_{ij}^s \tilde{f}_s^{ab} - f_{is}^a \tilde{f}_j^{sb} + f_{is}^b \tilde{f}_j^{sa} - f_{js}^b \tilde{f}_i^{sa} + f_{js}^a \tilde{f}_i^{sb} = 0. \tag{1}$$

In fact, similarly to a Lie group being determined up to some global features by its Lie algebra, a PLG is in one to one correspondence with a *Lie bialgebra*

(LBA). This is a pair (g, g^*) of Lie algebras dual as vector spaces whose structure constants satisfy (1). Note that the LBA structure is symmetric between g and g^*, so to each LBA we can associate a pair of PLGs G and G^*.

An equivalent definition of a LBA is given in terms of the cocommutator δ the dual of the $[,]^*$ commutator

$$\delta : g \to \wedge^2 g,\ \langle \delta(x), \xi \wedge \eta \rangle = \langle x, [\xi, \eta]^* \rangle, x \in g, \xi, \eta \in g^*.$$

Jacobi for $[,]^*$ implies co-Jacobi $(\delta \otimes id) \circ \delta = 0$. The compatibility condition (1) translates into the cocycle condition

$$\delta([x, y]) = [\Delta(x), \delta(y)] + [\delta(x), \Delta(y)]$$

where $\Delta(x) = x \otimes 1 + 1 \otimes x$ and similarly for y.

A *quasi-triangular Lie bialgebra* is a LBA such that there exists a $r \in g \otimes g$ which, for all $x \in g$ satisfies:

1. $\delta(x) = [r, \Delta(x)]$;
2. $I = r + \sigma(r)$ is adjoint invariant $[I, \Delta(x)] = 0$. Here σ is the permutation operator;
3. $(\delta \otimes id)r = [r_{13}, r_{23}]$, $(id \otimes \delta)r = [r_{13}, r_{12}]$.

A *factorizable Lie bialgebra* is a quasi-triangular LBA such that I is non-degenerate. One can use I to identify g and g^*. The factorization refers to the fact that any $x \in g$ can be decomposed as $x = x_+ - x_-$. Here

$$x_+ = \langle r, \xi \otimes id \rangle,\ x_- = -\langle r, id \otimes \xi \rangle$$

for some $\xi \in g^*$ satisfying $x = \langle I, \xi \otimes id \rangle$. Such a ξ always exists since I is non-degenerate.

A PLG G is quasi-triangular if its tangent LBA g is quasi-triangular. Similarly a PLG is factorizable if its tangent LBA is factorizable.

One can define two important Poisson brackets $\{,\}_\pm$ on a quasi-triangular LBA.

$$\{f,h\}_\pm = \langle r, \nabla f \otimes \nabla h\rangle \pm \langle r, \nabla' f \otimes \nabla' h\rangle \tag{2}$$

where

$$\langle \nabla f(x), \xi\rangle \equiv \frac{d}{dt} f(e^{t\xi}x),\ \langle \nabla' f(x), \xi\rangle \equiv \frac{d}{dt} f(xe^{t\xi})$$

are the left and right gradients respectively. The $\{,\}_-$ Poisson bracket makes G into a PLG. I will denote it simply by $\{,\}$. The other bracket $\{,\}_+$ is also very important since it is non-degenerate almost everywhere and makes G into a symplectic manifold.

For every representation ρ one can explicitly write the Poisson relations for the matrix elements of $T(x) = \rho(x)$ which are coordinates on the group as

$$\{T_1, T_2\} = [r_+, T_1 T_2] \tag{3}$$

where $r_+ = (\rho \otimes \rho) r$ and the subscript specifies the position in the tensor product. It is also useful to define $r_- = -(\rho \otimes \rho)\sigma(r)$.

The standard example of a factorizable PLG is $SL(N,C)$. In this case

$$r = \frac{1}{2} \sum_{i,j=1}^{N-1} (A^{-1})_{ij}\, H_i \otimes H_j + \sum_{i<j} E_{ij} \otimes E_{ji}$$

where A is the Cartan matrix, H_i are Cartan generators and E_{ij} are generators which in the fundamental representation are represented by matrices with only one non-vanishing entry equal to one in the ij position. In this case we can give an explicit description of the dual group $SL(N,C)^*$ and its Poisson structure despite the fact that it is not quasi-triangular. Let $SL(N,C)^*$ be the group of pairs of upper and lower triangular matrices $\{(L^+, L^-)\}$ where

$$L^+ = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_n \end{pmatrix},\quad L^- = \begin{pmatrix} a_1^{-1} & & 0 \\ & \ddots & \\ * & & a_n^{-1} \end{pmatrix},\quad \prod_{i=1}^{N} a_i = 1. \tag{4}$$

The group multiplication is given by multiplying corresponding matrices within each pair. Using the same notation for matrix group elements and functions on the group, the Poisson brackets are:

$$\begin{aligned} \{L_1^+, L_2^+\} &= [r_\pm, L_1^+ L_2^+], \\ \{L_1^-, L_2^-\} &= [r_\pm, L_1^- L_2^-], \\ \{L_1^+, L_2^-\} &= [r_+, L_1^+ L_2^-]. \end{aligned} \tag{5}$$

One can also define

$$L = (L^-)^{-1} L^+$$

and the Poisson brackets above become

$$\{L_1, L_2\} = L_1 r_+ L_2 + L_2 r_- L_1 - r_+ L_1 L_2 - L_1 L_2 r_-. \tag{6}$$

The derivation of this bracket from the quantum commutation relations is discussed in the appendix. The map from (L^+, L^-) to L is not one to one. It is a 2^{N-1} cover. Later we will define reality structures on this Poisson algebras.

Now I will give a more detailed description of the $SL(2,C)$ and $SL(2,C)^*$ groups. Let

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad L = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

The classical r-matrices can be written as 4×4 matrices

$$r_+ = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & -1/4 & 1 & 0 \\ 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}, \quad r_- = \begin{pmatrix} -1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & -1 & 1/4 & 0 \\ 0 & 0 & 0 & -1/4 \end{pmatrix}.$$

Using (3) after some algebra one obtains

$$\begin{aligned} \{a, b\} &= ab/2, \\ \{a, c\} &= ac/2, \end{aligned} \tag{7}$$

$$
\begin{aligned}
\{a,d\} &= cd, \\
\{b,c\} &= 0, \\
\{b,d\} &= bd/2, \\
\{c,d\} &= cd/2.
\end{aligned}
$$

Similarly using (6) one obtains

$$
\begin{aligned}
\{\alpha,\beta\} &= \alpha\beta, \\
\{\alpha,\gamma\} &= -\alpha\gamma, \\
\{\alpha,\delta\} &= 0, \\
\{\beta,\gamma\} &= \alpha(\alpha-\delta), \\
\{\beta,\delta\} &= \alpha\beta, \\
\{\gamma,\delta\} &= -\alpha\gamma.
\end{aligned} \tag{8}
$$

A further decomposition of L^+ as a diagonal matrix and an upper diagonal matrix with unit entries on the diagonal, and of L^- as a diagonal matrix and a lower diagonal matrix with unit entries on the diagonal, is possible. For the $SL(2,C)^*$ case, we have

$$
L^+ = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & \chi_+ \\ 0 & 1 \end{pmatrix}, \quad L^- = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\chi_- & 1 \end{pmatrix}.
$$

It corresponds to Gauss's decomposition of L

$$
L = \begin{pmatrix} 1 & 0 \\ \chi_- & 1 \end{pmatrix} \begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix} \begin{pmatrix} 1 & \chi_+ \\ 0 & 1 \end{pmatrix}.
$$

To every LBA (g, g^*) we can associate a factorizable LBA called the *double Lie bialgebra* (d, d^*). First we define $d = g + g^*$, i.e. the direct sum of vector spaces. It has a natural bilinear form $\langle , \rangle_d$ defined in terms of the dual pairing $\langle , \rangle$ of g and g^*

$$
\langle (x,\xi),(y,\eta)\rangle_d \equiv \langle x,\eta\rangle + \langle y,\xi\rangle, x,y \in g,\ \xi,\eta \in g^*.
$$

We define on d the unique Lie algebra such that:

1. g and g^* are subalgebras;

2. the bilinear form $\langle,\rangle_d$ determined by the dual pairing is adjoint invariant.

On the basis of d given by $\{e_i, e^i\}$, the commutator $[,]_d$ has the form

$$
\begin{aligned}
&[e_i, e_j]_d = f_{ij}^k \; e_k, \\
&[e^i, e^j]_d = f_k^{ij} \; e^k, \\
&[e^i, e_j]_d = f_{ik}^i \; e^k - \tilde{f}_{ik}^j \; e_k.
\end{aligned}
$$

Also $d^* \equiv g^* \oplus g$, i.e. it is the direct sum of Lie algebras $[e^i, e_j]_{d^*} = 0$. The pair (d, d^*) is a factorizable LBA with $r_d \equiv e^i \otimes e_i \in d \otimes d$, thus it is a projector on the g factor. Note that $sl(N, C)$ is almost the double of one of its Borel subalgebras[¶]. We can exponentiate d to a Lie group D and $\{,\}_-$ will make it into a PGL.

The simplest example of the above structure is obtained if we start from the trivial LBA (g, g^*), i.e. g is a Lie algebra and g^* its dual with the trivial commutator. G is a Lie group with Lie algebra g and $G^* = g^*$ is an abelian group. D is the cotangent bundle $T^*G = G \times g^*$. The $\{,\}_+$ bracket is the canonical Poisson bracket on the cotangent bundle, and $\{,\}_-$ is the Lie bracket on g^* extended by left translations to the cotangent bundle.

The double D of a factorizable PLG G can be described in more detail. As a group it is isomorphic with $G \times G$[‖]. The groups G and G^* are subgroups of D and are embedded as follows

$$G \subset G \times G, \quad T \to (T, T),$$

$$G^* \subset G \times G, \quad L \to (L^+, L^-).$$

[¶]It is the double of a Borel subalgebra divided by the Cartan subalgebra.

[‖]This is only true for complex groups. If G has a reality structure the double is obtain by imposing a reality structure on $G^c \times G^c$ where G^c is the complexification of G.

Almost all elements (x,y) of the double can be written in factorized form

$$(x,y) = (T,T)^{-1}(L^+,L^-) = (\tilde{L}^+,\tilde{L}^-)^{-1}(\tilde{T},\tilde{T}). \tag{9}$$

A pair of Poisson manifolds (P,P') is called a *dual pair* [12, 5] if there exists a symplectic manifold S and two projections π and π'

$$\begin{array}{ccccc} & & S & & \\ & \pi \swarrow & & \searrow \pi' & \\ P & & & & P' \end{array}$$

such that the sets of functions which are pullbacks of functions on P and P' centralize each other

$$\{\pi^*(f), \pi'^*(f')\}_S = 0,$$

An important theorem [12, 4] states that each symplectic leaf of P is obtained by projecting on P the preimage of an element a of P'

$$\pi(\pi'^{-1}(a)), \quad a \in P'.$$

The manifolds D/G and $G \setminus D$ form a dual pair. The symplectic manifold is the double D of G with the $\{,\}_+$ bracket. The following projections

$$\begin{array}{ccccc} & & D & & \\ & \pi \swarrow & & \searrow \pi' & \\ G \setminus D & & & & D/G \end{array}$$

can be used to induce Poisson structures on D/G and $G \setminus D$. Since D is factorizable $G^* \cong G \setminus D$. Moreover the Poisson structure induced on $G \setminus D$ from D coincides with the original Poisson structure on G^*. Then the above theorem gives the symplectic leaves of G^*. In particular if G is factorizable, $\pi'(x,y) \equiv xy^{-1} = a$ and the preimage of a has elements of the form (ay,y). Then $\pi(x,y) = y^{-1}x = y^{-1}ay$, thus the symplectic leaves are given by the orbits of the coadjoint action of G on $G \setminus D$. This action is also known as the dressing action [4]

$$G \times (G \setminus D) \to G \setminus D, \ (y,a) \to y^{-1}ay.$$

3 Symplectic Leaves

In the first part of this section, I will discuss the $SL(N,C)$ case. So far, everything was complex. The simplest reality structure one can impose is to require everything to be real. We then obtain $SL(N,R)$, its double, dual etc. However, we want to obtain $SU(N)$. We start on the double with the reality structure

$$x^\dagger = y^{-1}.$$

Since G and G^* are subgroups, this induces the following reality structures

$$T^\dagger = T^{-1},\ (L^+)^\dagger = (L^-)^{-1}. \tag{10}$$

Once we impose (10) the dual group is no longer simply connected, since a_i in (4) are real and non-zero. Define $SU(N)^*$ as the component connected to the unit element of the group.

$$SU(N)^* = \{(L^+, L^-) \in SL^*(N,C) \mid (L^+)^\dagger = (L^-)^{-1},\ a_i > 0\}.$$

We can also describe $SU(N)^*$ in terms of L as the set of hermitian, positive definite matrices of determinant one. Then the map $(L^+, L^-) \to L = (L^-)^{-1}L^+$ is one to one and the factorization is unique.

For $SU(2)^*$ the reality structure is $\bar{\alpha} = \alpha$, $\bar{\delta} = \delta$, $\bar{\beta} = \gamma$.

To summarize, the double of $SU(N)$ is $SL(N,C)$, and the factorization (9) can be written $x = T^{-1}L^+$, that is to say, any matrix of determinant one can be decomposed uniquely as the product of a special unitary matrix and an upper triangular matrix with real positive diagonal entries**.

In particular the double of $SU(2)$ is the proper Lorentz group $SL(2,C)$. It is interesting to note that the double of the trivial PLG $SU(2)$, i.e. its cotangent bundle, is the proper homogeneous Galilean group.

Using the two factorizations

$$(x,y) = (T^{-1}L^+, T^{-1}L^-) = ((\tilde{L}^+)^{-1}\tilde{T}, (\tilde{L}^-)^{-1}\tilde{T})$$

**Note that y is not independent $y = (x^\dagger)^{-1}$

and the projections $\pi(x,y) = y^{-1}x$, $\pi'(x,y) = xy^{-1}$ we obtain the following form for the symplectic leaves

$$\pi(\pi'^{-1}((\tilde{L}^+)^{-1}\tilde{L}^-)) = \{(L^-)^{-1}L^+ = (\tilde{T})^{-1}\tilde{L}^-(\tilde{L}^+)^{-1}\tilde{T} | \tilde{T} \in SU(2)\}$$

where $(\tilde{L}^+, \tilde{L}^-) \in SU(2)^*$ is fixed, and $\tilde{T}$ parametrizes the leave. This is just the orbit of the right Poisson coadjoint action of $SU(2)$ on $SU(2)^*$

$$L \to T^{-1}LT.$$

It is convenient to use an exponential parametrization of $L = (L^-)^{-1}L^+$

$$L = \exp(x_i\sigma_i) = \cosh(r) + \sinh(r) \begin{pmatrix} n_3 & n_- \\ n_+ & -n_3 \end{pmatrix}$$

where σ_i's are the Pauli matrices, $r^2 = \sum_i x_i^2$ and $n_i = x_i/r$. Since $tr(L) = 2\cosh(r)$ is invariant under the coadjoint action we see that the simplectic leaves are spheres of radius r except for the $r = 0$ leaf, which is zero dimensional. In terms of the exponential parametrization, the Poisson algebra (8) becomes

$$\{x_\pm, x_3\} = \pm x_\pm(x_3 + r\coth(r)),$$

$$\{x_-, x_+\} = 2x_3(x_3 + r\coth(r)).$$

Since r is constant on symplectic leaves it must be central in the above Poisson algebra, which can be checked by direct computation. These Poisson spheres and their quantization were first studied in [11]. One can parametrize the radius r sphere using stereographic projection coordinates $z, \bar{z}$

$$z = \frac{x_-}{r - x_3}, \ \bar{z} = \frac{x_+}{r - x_3}.$$

After some straightforward algebra we obtain

$$\{\bar{z}, z\}_r = \frac{1}{2}(1 + z\bar{z})^2 \left(\frac{z\bar{z} - 1}{z\bar{z} + 1} + \coth(r)\right).$$

The right action of $SU(2)$ on z by fractional transformations

$$z' = \frac{\bar{a}z - b}{\bar{b}z + a}$$

is a Poisson action i.e. a, b, c, d have non-trivial bracket given by (7). Since our path integral is formulated in real time, we do a Wick rotation and obtain the Minkowski Poisson bracket

$$\{\bar{z}, z\}_r = \frac{i}{2}\left(1 + z\bar{z}\right)^2 \left(\frac{z\bar{z} - 1}{z\bar{z} + 1} + \coth(r)\right) \tag{11}$$

differing from the original one by a phase factor.

Using non-singular coordinates around the south pole $w = -1/z$ the Poisson bracket becomes

$$\{\bar{w}, w\}_r = \frac{i}{2}\left(1 + w\bar{w}\right)^2 \left(-\frac{w\bar{w} - 1}{w\bar{w} + 1} + \coth(r)\right)$$

thus the Poisson structure is not north-south symmetric. The infinite r limit is singular at the south pole. This particular Poisson structure and its quantization was studied in [6, 7].

The small r limit is dominated by the $\coth(r)$ term and

$$\{\bar{z}, z\}_r \approx \frac{i}{2}\coth(r)\left(1 + z\bar{z}\right)^2. \tag{12}$$

This is the standard Poisson bracket on a sphere of radius $\coth^{1/2}(r)$. The right action by fractional transformations on (12) leaves this Poisson bracket invariant. Thus the small radius symplectic leaves are almost rotationally invariant.

Next we obtain the symplectic form on the leaves. Let f, h be functions on the leaf; each f defines a vector field v_f such that $v_f(h) = \{f, h\}$. Then the symplectic form is defined by

$$\Omega(v_f, v_h) \equiv \{h, f\}.$$

In local coordinates, the Poisson bracket and the symplectic form have the form

$$\{f, h\} = P^{ij}\, \partial_i f\, \partial_j h,\ \ \Omega = \frac{1}{2}\Omega_{ij} dx^i \wedge dx^j,$$

and the two antisymmetric tensors satisfy

$$P^{ij}\Omega_{jk} = \delta^i_k.$$

In complex coordinates, this is simply $P^{\bar{z}z}\Omega_{z\bar{z}} = 1$, and gives

$$\Omega = -\frac{2}{i}\frac{d\bar{z} \wedge dz}{(1+z\bar{z})^2}\left(\frac{z\bar{z}-1}{z\bar{z}+1} + \coth(r)\right)^{-1} = -\frac{\Omega_0}{n_3 + \coth(r)},$$

where Ω_0 is the standard area 2-form on the unit sphere.

4 Path Integral Quantization

The path integral quantization of the Poisson algebra on the leaves of $su(2)^*$ was discussed in [1, 2]. Quantization of these leaves gives the unitary representations of $SU(2)$. We will do the same for the symplectic leaves above and obtain the unitary representations of $\mathcal{U}_q(su(2))$ algebra. This is in fact a Hopf algebra but we concentrate here on the algebra structure[††].

Before starting the quantization we have to find canonical coordinates on the leaves. Note that

$$\Omega_0 = \sin\theta\, d\theta \wedge d\phi = d(-\cos(\theta)) \wedge d\phi$$

thus $(-\cos(\theta), \phi)$ are Darboux coordinates on the standard S_2. Similarly

$$\Omega = d[-\ln(n_3 + \coth(r))] \wedge d\phi$$

so we define

$$J \equiv -\ln\left[\frac{n_3 + \coth(r)}{(\coth^2(r) - 1)^{1/2}}\right] = -\ln\left[\cosh(r) + \sinh(r)\, n_3\right]$$

[††]The coproduct and antipode of the $L^{\pm}$ generators are the same as in the classical Poisson-Hopf algebra

where the denominator was fixed by the requirement that J spans a symmetric interval $(-r, r)$. We have $\Omega = dJ \wedge d\phi = d(J\, d\phi)$ so we define the Poincare 1-form Θ

$$\Theta = J\, d\phi + c\, d\phi$$

where c is a constant to be fixed later. Thus the Poisson sphere of radius r is parametrized by J and ϕ as

$$n_3 = \sinh^{-1}(r)(e^{-J} - \cosh(r)), \;\; n_\pm = (1 - n_3^2)^{1/2}\; e^{\mp i\phi}.$$

The Poisson algebra on any leaf can be quantized, but in general these quantum algebras will not have unitary representations. Unitarity leads to a quantization of the radius of the Poisson sphere. Before starting the Poisson case let us review two different quantization conditions used in [1, 2] for the trivial case. In [2] a geometric quantization condition similar to that used for the Dirac monopole or the Wess-Zumino-Witten model was used to obtain the allowed values of the radius. The action must be continuous as the path crosses over the poles. Equivalently

$$e^{i/\hbar \oint \Theta} = 1 \tag{13}$$

where the integral is over an infinitesimal loop around the poles. However this condition was only used to determine the characters of the representations. Also note that, unlike the Dirac monopole where the action is a configuration space action, both in the trivial and the Poisson case one has a phase space action.

However in [1] it was shown that in order to obtain the matrix elements of $su(2)$ a non-trivial phase has to exist as the path crosses the poles. Requiring the correct matrix elements one obtains the quantization condition

$$e^{i/\hbar \oint \Theta} = -1 \tag{14}$$

This gives the same result as (13) for the Cartan generator and thus for the characters. Here I will use (14) and show that we obtain the standard matrix elements of the quantum qroup generators.

Imposing (14) at the north and south poles we obtain the quantization $r = N\hbar/2$ where N is a positive integer. For N odd one can set $c = 0$ but a non-zero c is required for even N. The simplest choice is $c = \hbar/2$. We can write the two cases together as

$$\Theta = (J + M\hbar/2)\, d\phi,\ \ M = 0, 1.$$

Next I list some of the functions on the Poisson sphere that I will quantize, expressed in terms of Darboux variables J, ϕ

$$\begin{aligned}
\alpha &= e^{-J} \\
\beta &= (-1 + 2\cosh(r)e^{-J} - e^{-2J})^{1/2} e^{i\phi} \qquad (15) \\
\gamma &= (-1 + 2\cosh(r)e^{-J} - e^{-2J})^{1/2} e^{-i\phi} \\
\delta &= 2\cosh(r) - e^{-J} \\
a &= e^{-J/2} \\
\chi_{\pm} &= (-1 + 2\cosh(r)e^{J} - e^{2J})^{1/2} e^{\pm i\phi}
\end{aligned}$$

The general structure of this functions is

$$\mathcal{O}(J, \phi) = \mathcal{F}(J) e^{ip\phi},\ \ p = 0, \pm 1.$$

Note also that

$$tr(L) = 2\cosh(r) = 2\cosh\left(N\hbar/2\right) = q^N + q^{-N},$$

where we introduced $q \equiv e^{\hbar/2}$. Since $tr(L)$ only depends on r, it is central in the Poisson algebra and will be central in the quantum algebra. In fact $tr(L)$ is the Casimir of $\mathcal{U}_q(su(2))$.

Next we discuss the Feynman path integral. Consider first for simplicity a Hamiltonian $H(J)$, i.e. a function of J and not of ϕ. Wave functions are functions on S_1 (or periodic functions of ϕ) and let $\mid \phi\rangle$ be a ϕ eigenvector. The propagator on S_1 can be expressed in terms of the propagator on the covering space of S_1, which is the real line by

$$\langle \phi' \mid e^{-\frac{i}{\hbar}HT} \mid \phi\rangle = \sum_{n\in Z} \langle \phi' + 2\pi n \mid e^{-\frac{i}{\hbar}HT} \mid \phi\rangle_0 \qquad (16)$$

where formally

$$\langle \phi' \mid e^{-\frac{i}{\hbar}HT} \mid \phi\rangle_0 = \int\!\!\int \frac{\mathcal{D}J\,\mathcal{D}\phi}{2\pi\hbar}\; e^{\frac{i}{\hbar}\int_0^T[\Theta - H(J)\,dt]} \tag{17}$$

where ϕ is integrated over the whole real line and J over the $(-r, r)$ interval. To make sense of the formal expression we divide T into P intervals and let $\phi_0 = \phi, \phi_P = \phi'$. Then

$$\langle \phi' \mid e^{-\frac{i}{\hbar}HT} \mid \phi\rangle_0 = \int \frac{\prod_i dJ_i}{2\pi\hbar} \int \prod_i d\phi_i\; e^{i/\hbar \sum_i [(J_i+c)(\phi_i-\phi_{i-1}) - H(J_i)T/P]} \tag{18}$$

The ϕ integration can be performed leading to delta functions which allow us to do all but one of the J integrals. Then the propagator on S_1 takes the form

$$\langle \phi' \mid e^{-\frac{i}{\hbar}HT} \mid \phi\rangle = \sum_{n\in Z} \int_{-N\hbar/2}^{N\hbar/2} \frac{dJ}{2\pi\hbar}\; e^{-i/\hbar H(J)T} e^{i/\hbar (J+c)(\phi'+2\pi n)} e^{-i/\hbar (J+c)\phi}$$

Using the Poisson resummation formula

$$\sum_{n\in Z} e^{2\pi i n\alpha} = \sum_{k\in Z} \delta(\alpha - k)$$

we perform the last integral and obtain

$$\langle \phi' \mid e^{-\frac{i}{\hbar}HT} \mid \phi\rangle = \sum_{\substack{k \\ |J_k|\le N\hbar/2}} \frac{e^{ik\phi'}}{\sqrt{2\pi}}\; e^{-i/\hbar H(J_k)T}\; \frac{e^{-ik\phi}}{\sqrt{2\pi}}$$

where $J_k = \hbar(k - M/2)$. The sum is over all integers k such that $(-N + M)/2 \le k \le (N+M)/2$. We see that not all states propagate. We can make the path integral unitary by projecting out the states that do not propagate. Define the Hilbert space as the vector space spanned by the vectors

$$\mid m\rangle = \int \frac{d\phi}{\sqrt{2\pi}}\; e^{i(m+M/2)\phi} \mid \phi\rangle, \quad m = -j, \ldots, j$$

where, according to angular momentum conventions, j is a half integer such that $N = 2j + 1$. Note that the exponent is always an integer and N is the total number of states. The maximum value $J = \pm N\hbar/2$ is not reached quantum mechanically. It differs from the results in [2] but agrees with [1] as previously mentioned. It was pointed out in [1] that this is similar to the non-zero ground state energy of the harmonic oscillator.

5 Matrix Elements and the Quantum Algebra

Since this is a phase space path integral some care must be taken when quantizing functions which depend on canonically conjugate variables. The standard mid-point prescription for a function of the form $\mathcal{J}(J)\Phi(\phi)$ is to write it as $\mathcal{J}(J_i)\Phi[(\phi_i + \phi_{i-1})/2]$ in the path integral. Thus for functions of the form $\mathcal{O}(J,\phi) = \mathcal{F}(J)e^{ip\phi}$ I will use $\mathcal{F}(J_i)e^{ip(\phi_i+\phi_{i-1})/2}$. To calculate the matrix elements of such an operator we insert it in the path integral (18) with $H = 0$ and take T infinitesimal. For the prescription above it is sufficient to consider only one time interval. The matrix elements are

$$\langle \phi' \mid \mathcal{O} \mid \phi \rangle = \sum_{n \in Z} \int \frac{dJ}{2\pi\hbar}\, e^{i/\hbar (J+c)(\phi' + 2\pi n - \phi)} \mathcal{F}(J) e^{ip(\phi' + 2\pi n + \phi)/2} =$$

$$\sum_k \frac{e^{ik\phi'}}{\sqrt{2\pi}}\, \mathcal{F}(J_k)\, \frac{e^{-i(k-p)\phi}}{\sqrt{2\pi}}$$

where $J_k = \hbar(k - M/2 - p/2)$, and I used Poisson resummation before performing the J integral. Then the matrix elements in the $\{\mid m\rangle\}$ basis are given by

$$(\mathcal{O})_{m'm} = \langle m' \mid \mathcal{O} \mid m \rangle = \mathcal{F}[(m' - p/2)\hbar]\, \delta_{m'-p-m,0}, \;\; m = -j, \ldots, j. \quad (19)$$

Using the opposite mid-point prescription $\mathcal{F}[(J_i + J_{i-1})/2]e^{ip\phi_i}$ gives the same matrix elements. However in this case one has to consider at least two time intervals if working in the ϕ representation. This prescription is more convenient when working in the J representation.

We can use (19) to calculate matrix elements of any function on $SU(2)^*$. Mid-point prescription in the path integral results in a special ordering of the quantum operators, when expressed in terms of J and ϕ, called Weyl ordering. If one starts from the Gauss's decomposition, uses path integral to obtain the matrix elements of a and $\chi_\pm$ and then uses them to express

$L^{\pm}$ as products of quantum matrices, we obtain the quantum commutation relations [9]. Using (19) we obtain

$$\begin{aligned} (a)_{m'm} &= e^{-\hbar m'/2}\,\delta_{m'-m,0} \\ (\chi_{\pm})_{m'm} &= (-1+2\cosh(\hbar(j+1/2))e^{\hbar(m'\mp 1/2)} - e^{2\hbar(m'\mp 1/2)})^{1/2}\,\delta_{m'-m\mp 1,0} \end{aligned} \tag{20}$$

One can check by direct calculation that relations (20) are representations of the algebra generated by $a, \chi_{\pm}$ with relations

$$\begin{aligned} \chi_+ a &= q a \chi_+ \\ \chi_- a &= q^{-1} a \chi_- \\ q\chi_+\chi_- - q^{-1}\chi_-\chi_+ &= \lambda(a^{-4}-1) \end{aligned} \tag{21}$$

where $\lambda \equiv q - q^{-1}$. Using this we define the quantum matrices $L^{\pm}$ as

$$L^+ = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & \chi_+ \\ 0 & 1 \end{pmatrix},$$

$$L^- = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\chi_- & 1 \end{pmatrix}.$$

One can use (21) to check that $L^{\pm}$ satisfies the quantum group commutations relations [8, 9, 10]

$$\begin{aligned} R_{\pm} L_1^{\pm} L_2^{\pm} &= L_2^{\pm} L_1^{\pm} R_{\pm} \\ R_+ L_1^+ L_2^- &= L_2^- L_1^+ R_+ \\ R_- L_1^- L_2^+ &= L_2^+ L_1^- R_- \end{aligned} \tag{22}$$

where the quantum matrices are given in the appendix. Alternatively, using the representations

$$L^+ = \begin{pmatrix} q^{-H/2} & q^{-1/2}\lambda X_+ \\ 0 & q^{H/2} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{H/2} & 0 \\ -q^{1/2}\lambda X_- & q^{-H/2} \end{pmatrix} \tag{23}$$

of the quantum $L^{\pm}$ in terms of Jimbo-Drinfeld generators discussed in the appendix, the relations (21) are equivalent to

$$[H, X_{\pm}] = \pm 2 X_{\pm},\ [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \tag{24}$$

The Jimbo-Drinfeld generators of $\mathcal{U}_q(su(2))$ can be obtained in the path integral by inserting

$$\begin{aligned} H &= \hbar^{-1}\, 2J \\ X_{\pm} &= \lambda^{-1}[2(\cosh(r) - \cosh(J)]^{1/2}\, e^{\pm i\phi}. \end{aligned} \tag{25}$$

Note that unlike a and $\chi_{\pm}$ the insertions above are already quantum. In addition while the functional dependence in terms of J and ϕ can be easily obtained from (23) the overall normalization of $X_{\pm}$ has been adjusted to give the standard result. The same kind of normalization adjustments are necessary if one tries to insert the matrix elements of $L^{\pm}$ directly into the path integral. This just reflects ordering ambiguities of quantum operators. Alternatively one could get the standard result without any adjustments of normalization by using a non-midpoint prescription. For example the off-diagonal element of L^+ equals $a\chi_+$ with this specific ordering in the quantum case. Since the path integral gives time ordering we can obtain the desired quantum ordering by using the following prescription

$$e^{-J_i/2}(-1 + 2cosh(r)e^{(J_i+J_{i-1})/2} - e^{J_i+J_{i-1}})^{1/2}e^{+i\phi}$$

Note that I only used a mid-point prescription for χ_+ and not for a. The matrix elements obtained using (19) are

$$\begin{aligned} (H)_{m'm} &= 2m\, \delta_{m'-m,0}, \\ (X_{\pm})_{m'm} &= \{2\coth[\hbar(j+1/2)] - 2\coth[\hbar(m \pm 1/2)]\}^{1/2}\, \delta_{m'-m\mp 1,0}. \end{aligned}$$

The generators of $su(2)$ are obtained using

$$\begin{aligned} \tilde{H} &= 2J, \\ \tilde{X}_{\pm} &= (r^2 - J^2)^{1/2}\, e^{\pm i\phi}. \end{aligned} \tag{26}$$

In this case it is possible to write all generators without using $\hbar$ while in the deformed case a different rescaling for each generator is required to eliminate $\hbar$. The matrix elements obtained using (19)

$$\begin{aligned}(\tilde{H})_{m'm} &= 2\hbar m\, \delta_{m'-m,0}, \\ (\tilde{X}_\pm)_{m'm} &= \hbar[(j+1/2)^2 - (m \pm 1/2)^2]^{1/2}\, \delta_{m'-m\mp 1,0}\end{aligned}$$

are just the standard matrix elements of the $su(2)$ algebra

$$[\tilde{H}, \tilde{X}_\pm] = \pm 2\hbar \tilde{X}_\pm, \ [\tilde{X}_+, \tilde{X}_-] = \hbar \tilde{H}.$$

6 Concluding Remarks

In addition to trying to generalize the results in [1, 2] my goal in this paper was to better understand the quantization (22) of the Poisson bracket (5). Any $R_\pm$ satisfying $R_\pm = 1 + \hbar r_\pm + \mathcal{O}(\hbar^2)$ used in (22) would give the same Poisson bracket in the classical limit. The $\mathcal{O}(\hbar^2)$ and higher order terms are fixed by requireing that (22) are commutation relations of a Hopf algebra deformation of the original Poisson-Hopf algebra. It is natural then to ask what is the relation of this quantization to the quantization known as Weyl quantization. Of course this question could be answered using algebraic methods without appealing to path integrals. At least for the case of $SU(2)$, I found that the functions $\chi_\pm$ and a appearing in the Gauss's decomposition play a special role. Their quantization using Weyl ordering gives the same commutation relations as in the quantum group quantization. It would be interesting to investigate if this result still holds for an arbitrary $SU(N)$.

It should be possible to generalize the path integral formulated in this paper to arbitrary classical groups. The similarity between the trivial and the Poisson cases for $SU(2)$ suggests that a starting point could be the path integral quantization of the coadjoint orbits of classical groups discussed in [2].

The existence of a non-trivial phase as the path crosses the poles discussed in [1] is present in the Poisson case too. A better understanding of the origin

of this phase would be welcomed.

Let us now compare the trivial and Poisson cases. The symplectic leaves in both cases are spheres parametrized by $(z, \bar{z})$ in stereographic projection. The group $SU(2)$ acts in the same way on the leaves in the two cases, i.e. by standard rotations of the spheres, but in the trivial case the bracket is invariant under the action, while in the Poisson case the action is only a Poisson action. However, once the symplectic form is expressed in Darboux coordinates (J, ϕ) the leaves appear to be identical. As a consequence the path integral has the same form as in [1, 2], but since the transformation to the Darboux variables is non-trivial in the Poisson case, $SU(2)$ acts in a complicated way on the leaves, and functions on $SU(2)^*$ have a complicated dependence on (J, ϕ). Compare for example (25) and (26). Thus the same path integral generates different matrix elements because we insert different functions in the trivial and Poisson cases. This shows explicitly that on the same symplectic manifold one can implement both a trivial and a Poisson symmetry. The question of which is the actual symmetry of the system is a dynamical one, and can only be answered after we know the Hamiltonian. Finally, I conjecture that as in the $SU(2)$ case, for an arbitrary classical group, the path integral has the same form in the trivial and Poisson cases.

Acknowledgements

I would like to thank Professor Bruno Zumino for many useful discussions and valuabe comments. I would also like to thank Paolo Aschieri and Harold Steinacker for valuable input. This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-90-14797.

Appendix

Here we list some relations defining the quantum group $Fun_q(SU(2)^*)$ and discuss its relation to $\mathcal{U}_q(su(2))$ [8, 9, 10]. We only discuss the algebra and ignore all other issues. The quantum qroup $Fun_q(SU(2)^*)$ is a factorizable quasi-triangular Hopf algebra. As an algebra it is generated by triangular matrices $L^{\pm}$ satisfying quantum commutation relations

$$\begin{aligned} R_{\pm} L_1^{\pm} L_2^{\pm} &= L_2^{\pm} L_1^{\pm} R_{\pm} \\ R_{+} L_1^{+} L_2^{-} &= L_2^{-} L_1^{+} R_{+} \\ R_{-} L_1^{-} L_2^{+} &= L_2^{+} L_1^{-} R_{-} \end{aligned} \tag{27}$$

where

$$R_+ = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \ R_- = q^{1/2} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}.$$

The universal enveloping algebra $\mathcal{U}_q(su(2))$ is a quasi-triangular Hopf algebra. It has generators $H, X_{\pm}$ which satisfy the Jimbo-Drinfeld relations

$$[H, X_{\pm}] = \pm 2X_{\pm}, \ [X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}. \tag{28}$$

In [9] it was shown that these two Hopf algebras are isomorphic. The isomorphism is given by

$$L^+ = \begin{pmatrix} q^{-H/2} & q^{-1/2}\lambda X_+ \\ 0 & q^{H/2} \end{pmatrix}, \ L^- = \begin{pmatrix} q^{H/2} & 0 \\ -q^{1/2}\lambda X_- & q^{-H/2} \end{pmatrix}. \tag{29}$$

As in the classical case we can define the matrix $L = (L^-)^{-1} L^+$. It satisfies the following equation:

$$R_+^{-1} L_1 R_+ L_2 = L_2 R_-^{-1} L_1 R_- \tag{30}$$

as can be checked using (22).

In the classical limit we define $r_\pm$ matrices by $R_\pm = 1 + \hbar r_\pm + \mathcal{O}(\hbar^2)$. Then

$$(1-\hbar r_+)L_1(1+\hbar r_+)L_2 = L_2(1-\hbar r_-)L_1(1+\hbar r_-) + \mathcal{O}(\hbar^2)$$

and we obtain the following Poisson structure

$$\{L_1, L_2\} \equiv \lim_{\hbar\to 0}\frac{[L_1, L_2]}{-\hbar} = +L_1 r_+ L_2 + L_2 r_- L_1 - r_+ L_1 L_2 - L_1 L_2 r_- .$$

This is just the original Poisson bracket (6) which was the starting point for the path integral quantization.

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